VERTEX COLORING OF A FUZZY GRAPH USING ALPA CUT

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Abstract

Let G= (V_F,E_F) be a simple connected undirected fuzzy graph where V_F is a fuzzy set of vertices where each vertices has membership value μ and E_F is a fuzzy set of edges where each edge has a membership value σ. Vertex coloring is a function which assigns colors to the vertices so that adjacent vertices receive different colors. In this paper, we introduce a coloring function of fuzzy graph (crisp mode) to color all the vertices of graph of G and find the chromatic number of graph G which is a fuzzy number. The function is based on α cut of graph G. For different value of α cut which is depended on edge and vertex membership value of graph G we will get different graph and different chromatic number.

Keywords: Fuzzy set, Fuzzy graph, α cut, Fuzzy number

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1 Introduction

Graph theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include biochemistry (DNA double helix and SNP assembly problem), Chemistry (model chemical compounds) electrical engineering (communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling).

Graph coloring is one of the most important concepts in graph theory and is used in many real time applications like Job scheduling [8] , Aircraft scheduling [8], computer network security[9], Map coloring and GSM mobile phone networks[10] Automatic channel allocation for small wireless local area networks[11].The proper coloring of a graph is the coloring of the vertices with minimal number of colors such that no two adjacent vertices should have the same color. The minimum number of colors is called as the chromatic number and the graph is called properly colored graph.

We know that graphs are simply model of relation. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. In many real world problem, we get partial information about that problem. So there is vagueness in the description of the objects or in its relationships or in both. To describe this type of relation,we need to design fuzzy graph model. Fuzzy graph coloring is one of the most important problems of fuzzy graph theory. It uses in combinatorial optimization like traffic light control[8], exam-scheduling[9], register allocation etc. Two types of coloring namely vertex coloring and edge coloring are usually associated with any graph.

The first definition of a fuzzy graph was by Kaufmann in 1973, based on Zadeh's fuzzy relations . But it was Azriel Rosenfeld [2] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. During the same time R.T.Yehand S.Y. Bang have also introduced various connectedness concepts in fuzzy graphs. The fuzzy vertex coloring of a fuzzy graph was defined by the authors Eslahchi and Onagh and another approach of vertex coloring of fuzzy graph was used in S. Munoz, T. Ortuno. In our paper, we determine fuzzy chromatic number of a fuzzy graph G whose edge and vertices both are fuzzy.

2Preliminary Notes

Preliminary notes are given in two subsections

In first section basic concepts of fuzzy set and fuzzy graph are discussed

Definition 2.1.1:A fuzzy set A defined on a non empty set X is the family $A = \{(x, \mu_A(x))/x \epsilon X\}$ where μA: $x \rightarrow I$ is the membership function. In fuzzy set theory the set I is usually defined as the interval [0,1] such that $\mu A(x)$ =0 if x does not belong to A, $\mu A(x) = 1$ if x strictly belongs to A and any intermediate value represents the degree in which x could belong to A. The set I could be discrete set of the form $I = \{0,1,\ldots k\}$ where $\mu_A(x) < \mu_A(x_1)$ indicates that the degree of membership of x to A is lower than the degree of membership of x_1.In this paper, we define vertex and edge both are fuzzy set.

Definition 2.1.2: α cut set of fuzzy set A is defined as A_α is made up of members whose membership is not less than α .$A_\alpha = \{x \epsilon X \mid \mu_A(x) \geq \alpha.\}$. α cut set of fuzzy set is crisp set. In this paper, α cut set depend on vertex and edge membership value.

Definition 2.1.3: Blue et al. have given five types of graph fuzziness .Fuzzy graph is a graph G_F satisfying one of the following types of fuzziness (G_F of the ith type) or any of its combination:

(i) $G_{F1} = \{ G_1 , G_2 , G_3 , . . . , G_F \}$ where fuzziness is on each graph G_i.

(ii) $G_{F2} = \{ V, E_F \}$ where the edge set is fuzzy.

iii) $G_{F3} = \{ V, E(t_F , h_F) \}$ where both the vertex and edge sets are crisp, but the edges have fuzzy heads $h(e_i)$ and fuzzy tails $t(e_{i.})$

iv) $G_{F4} = \{ V_F , E \}$ where the vertex set is fuzzy.

(v) $G_{F5} = \{V, E(w_F) \}$ where both the vertex and edge sets are crisp but the edges have fuzzy weights.

In this paper, we use a fuzzy graph G which is a combination of G_{F2} and G_{F4}. So fuzzy graph $G= G_{F2} \cup G_{F4}$. We can define this fuzzy graph using their membership value of vertices and edges. Let V be a finite nonempty set. The triple $G= (V, \sigma, \mu)$ is called a fuzzy graph on V where μ and σ are fuzzy sets on V and E $(V \times V)$, respectively, such that $\mu(\{u, v\}) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$.

Note that a fuzzy graph is a generalization of crisp graph in which

$\mu(v) = 1$ for all $v \in V$

and $\rho(i, j) = 1$ if $(i, j) \in E$

$=0$ otherwise

so all the crisp graph are fuzzy graph but all fuzzy graph are not crisp graph.

Definition 2.1.4: The α cut of fuzzy graph defined as $G_\alpha = (V_\alpha, E_\alpha)$ where $V_\alpha = \{v \in V | \sigma \geq \alpha\}$ and $E_\alpha = \{e \in E | \mu \geq \alpha\}$.

Definition 2.1.5: Two vertices u and v for any strong edge in G are called adjacent if $(1/2)\min\{\sigma(u), \sigma(v)\} \leq \mu(uv)$. The degree of vertex v in G, denoted by $\deg_G v$, is the number of adjacent vertices to v and the maximum degree of G is defined by $\Delta(G) = \max\{\deg_G v \mid v \in V\}$.

Definition 2.1.6: Two edges v_iv_j and v_jv_k are said to be incident if $2\min\{\mu(v_iv_j), \mu(v_jv_k),\} \leq \sigma(v_j)$ for $j = 1,2\ldots|v|$, $1 \leq i, k \leq |v|$.

Second section we will define some basic concept of coloring of graph and coloring of fuzzy graph those are used in this paper.

Definition 2.2.1: Given a graph $G=(V,E)$, a coloring function is a mapping $C: V \rightarrow N$ identifying $C(i)$ as the color of node $i \epsilon V$, in such a way that two adjacent nodes cannot share the same color, i.e., $C(i) \neq C(j)$ if $\{i,j\} \epsilon E$. These nodes i and j will be denoted as incompatible and, in this context, graph G will be denoted the incompatibility graph. A k−coloring $C_k$is a coloring function with no more than k different colors $C_k: V \rightarrow 1,.K$ A graph is k−colored if it admits a k−coloring. The minimum value k such that G is k-colored is the chromatic number of G and it is denoted as $\chi(G)$. The graph coloring problem (for short, the coloring problem) consists of determining the chromatic number of a graph and an associated coloring function. This problem is known to be NP-hard.

The fuzzy vertex coloring of fuzzy graph was defined by the authors Eslahchi and Onagh .They defined chromatic number as the least value of k for which the fuzzy graph G has k-fuzzy coloring and k-fuzzy coloring is defined as follows

Definition 2.2.2:A family of fuzzy sets on V is called a k-coloring of fuzzy graph G= (V,σ,μ)

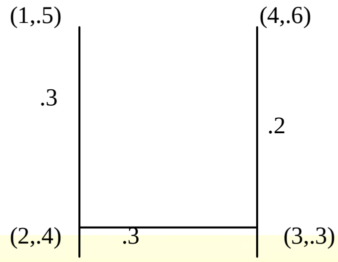
(a) $\vee \Gamma = \sigma$,

(b) $\gamma_1 \wedge \gamma_2 \; {}_{=}0$

(c)for every strong edge xy of G ,min{(γ_1(x), γ_2(y)}=0

(1≤i≤k)

Example:



The chromatic number of fuzzy graph G is 2

$\gamma_1(u_i)$=.5 i=1
.3 i=3
0 0therwise

$\gamma_2(u_i)$= .6 i=2
.4 i=4
0 0therwise

vertices	γ_1	γ_2	**max**
1	.5	0	.5
2	0	.6	.6
3	.3	0	.3
4	0	.4	.4

3 Coloring function of fuzzy graph (crisp mode)

3.1Definition: Given a fuzzy graph G=(V_F,E_F),its chromatic number is fuzzy number is chromatic number is fuzzy number $\chi(G)=\{(x_\alpha ,\alpha)\}$ where x_α is the chromatic number of G_α and α values are the different membership value of vertex and edge of graph G.

In this paper, we use α values are all different membership value of vertex and edge of fuzzy graph G. We find the all graph G_α which is a crisp graph for all α. Then we find minimum number of color needed to color the graph G_α . In such way, we find the fuzzy chromatic number which is a fuzzy number is calculated by its α cut . In 3.2 section we take a fuzzy graph G and describe the coloring function of fuzzy graph(crisp mode)

3.2 Example Let we take a fuzzy graph G=(V,σ,μ) where V has five vertices and membership value of those vertices are σ={0.9,0.7,0.8,0.7,1 and graph G has 10 edge. Membership value of those edges are in μ

$$\mu=\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0.0 & 0.6 & 0.8 & 0.7 & 0.5 \\ v_2 & 0.6 & 0.0 & 0.6 & 0.5 & 0.6 \\ v_3 & 0.8 & 0.6 & 0.0 & 0.7 & 0.8 \\ v_4 & 0.7 & 0.5 & 0.7 & 0.0 & 0.0 \\ v_5 & 0.5 & 0.6 & 0.8 & 0.5 & 0.0 \end{array}$$

$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & e_{10} & e_7 & e_5 & e_1 \\ v_2 & e_{10} & 0 & e_9 & e_6 & e_8 \\ v_3 & e_7 & e_9 & 0 & e_3 & e_4 \\ v_4 & e_5 & e_6 & e_3 & 0 & e_2 \\ v_5 & e_1 & e_8 & e_4 & e_2 & 0 \end{array}$$

Adjacent matrix 1 Adjacent matrix 2

Adjacent matrix 1 represent the membership value of edges and Adjacent matrix2 represent the name of the edge between the vertices.

Example :Edge between two vertices v_1 and v_2 is e_{10} and membership value of that edges 0.6.

Let fuzzy graph $G=(V_F,E_F)$ be a fuzzy graph where $V_F=\{(v_1,0.9),(v_2,0.7),(v_3,0.8),(v_4,0.7),(v_5,1)\}$and $E_F=\{(e_1,0.5)(e_2,0.5)(e_3,0.7)$ $(e_4,0.8)$ $(e_5,0.7)$ $(e_6,0.5)(e_7,0.8)(e_8,0.6)(e_9,0.6)(e_{10},0.6)\}$.

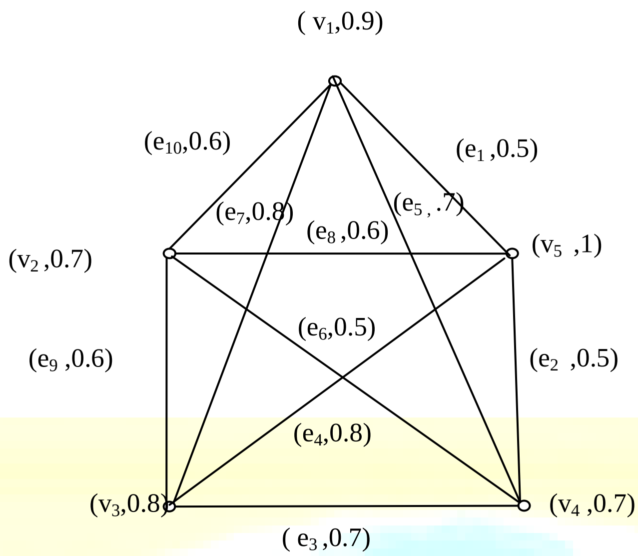


Figure 1

In this fuzzy graph, there are the six α cut is presented. There are {.5,.6,.7,.8,.9,1}.For every value of α ,we find graph G_α and find its fuzzy chromatic number.

For α=.5 Fuzzy graph G =(V,σ,μ) where σ={0.9,0.7,0.8,0.7,1} and

$$\mu = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0.0 & 0.6 & 0.8 & 0.7 & 0.5 \\ 0.6 & 0.0 & 0.6 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.0 & 0.7 & 0.8 \\ 0.7 & 0.5 & 0.7 & 0.0 & 0.0 \\ 0.5 & 0.6 & 0.8 & 0.5 & 0.0 \end{bmatrix} \end{array}$$

For α=.5 $G_{.5}=(V_{.5}, E_{.5})$where $V_{.5}$ and $E_{0.5}$are both are crisp set.$V_{0.5}$ is collection of those element of fuzzy set V_F whose membership value is greater than or equal to 0.5.So $V_{.5}=\{v_1,v_2,v_3, v_4, v_5\}$and same as $E_{.5}$ is collection of those element of fuzzy set E_F whose membership value is greater than or equal to 0.5. $E_{.5} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, .e_9, e_{10}\}$. Here we need minimum 5 color to proper color all the vertices of the graph $G_{.5}$. so the chromatic number of $G_{.5}$ is 5

For α=.5 ⇒ $\chi_{.5} = \chi(G_{.5}) = 5$

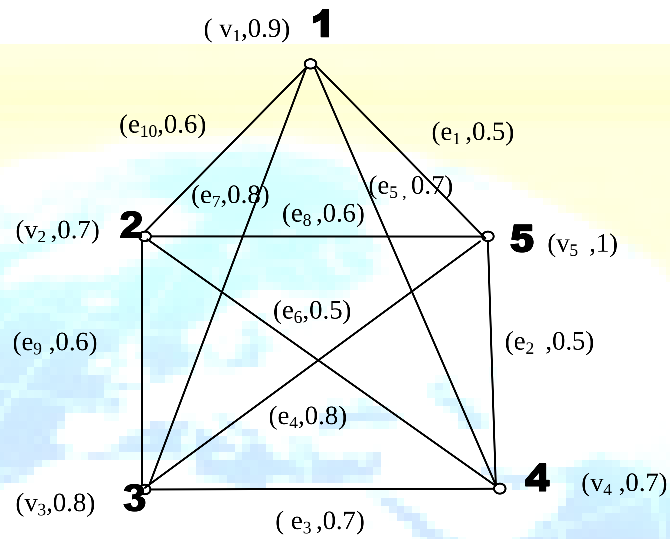


Figure 2

For α=.6, Fuzzy graph G =(V,σ,μ)where has σ={0.9,0.7,0.8,0.7,1} and

$$\mu = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0.0 & 0.6 & 0.8 & 0.7 & 0.0 \\ 0.6 & 0.0 & 0.6 & 0.0 & 0.6 \\ 0.8 & 0.6 & 0.0 & 0.7 & 0.8 \\ 0.7 & 0.0 & 0.7 & 0.0 & 0.0 \\ 0.0 & 0.6 & 0.8 & 0.0 & 0.0 \end{bmatrix} \end{array}$$

we find the graph $G_{.6} = (V_{0.6}, E_{0.6})$ where $V_{0.6} = \{v_1, v_2, v_3, v_4, v_5\}$ and $E_{0.5} = \{e_3, e_4, e_5, e_6, e_7, e_8, .e_9, e_{10}\}$ Here we need minimum 3color to proper color all the vertices of the graph $G_{.6}$. so the chromatic number of $G_{.6}$ is 3.

For $\alpha = 0.6 \Rightarrow \chi_{0.6} = \chi(G_{0.6}) = 3$

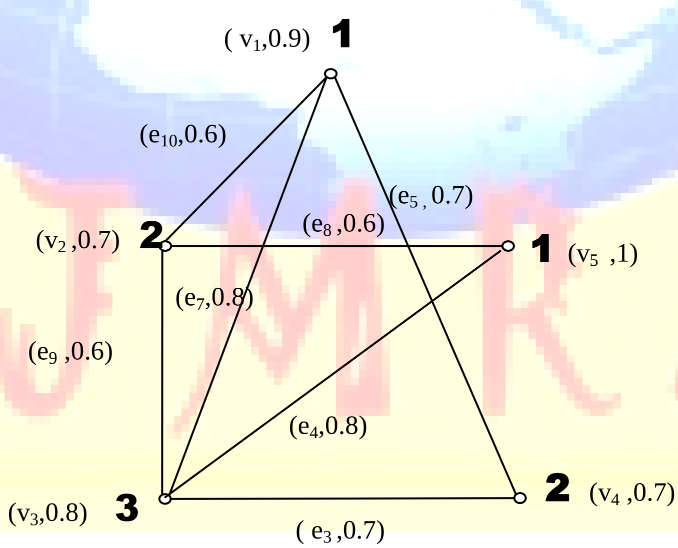


Figure 3

For $\alpha = .7$, Fuzzy graph $G = (V, \sigma, \mu)$ where $\sigma = \{0.9, 0.7, 0.8, 0.7, 1\}$ and

$$\mu = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0.0 & 0.0 & 0.8 & 0.7 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.8 & 0.0 & 0.0 & 0.7 & 0.8 \\ 0.7 & 0.0 & 0.7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 & 0.0 & 0.0 \end{bmatrix} \end{array}$$

we find the graph $G_{.7} = (V_{.7}, E_{.7})$ where $V_{.7} = \{v_1, v_2, v_3, v_4, v_5\}$ and $E_{.7} = \{e_3, e_4, e_5, e_7, e_8,\}$ Here we need minimum 3color to proper color all the vertices of the graph $G_{.7}$. so the chromatic number of $G_{.7}$ is .3.

For $\alpha = .7 \Rightarrow \chi_{.7} = \chi(G_{.7}) = 3$

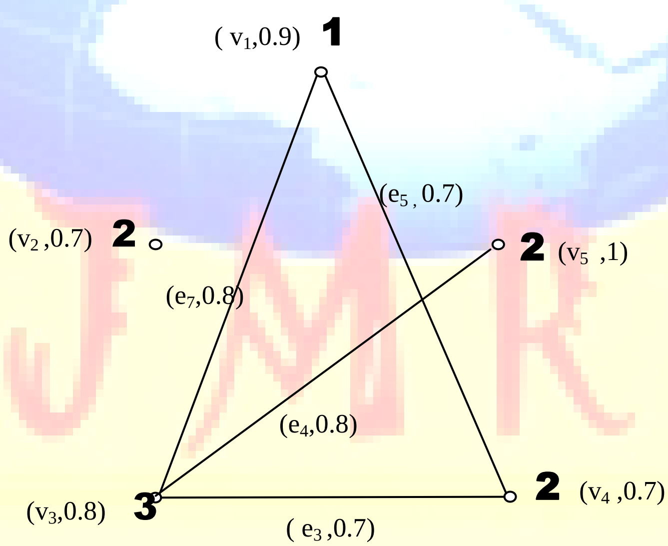


Figure 4

For $\alpha = .8$, Fuzzy graph $G = (V, \sigma, \mu)$ where $\sigma = \{0.9, 0.8, 1\}$ and

$$\mu = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_3 & v_5 \end{array} \\ \begin{bmatrix} 0.0 & 0.8 & 0.0 \\ 0.8 & 0.0 & 0.8 \\ 0.0 & 0.8 & 0.0 \end{bmatrix} \end{array}$$

we find the graph $G_{.8} = (V_{.8}, E_{.8})$ where $V_{.8} = \{v_1, v_3, v_5\}$ and $E_{.8} = \{e_4, e_7\}$.Here we need minimum 2color to proper color of all the vertices of the graph $G_{.6}$. so the chromatic number of $G_{.8}$ is 2. For $\alpha=.8 \Rightarrow \chi_{.8} = \chi(G_{.8})=2$

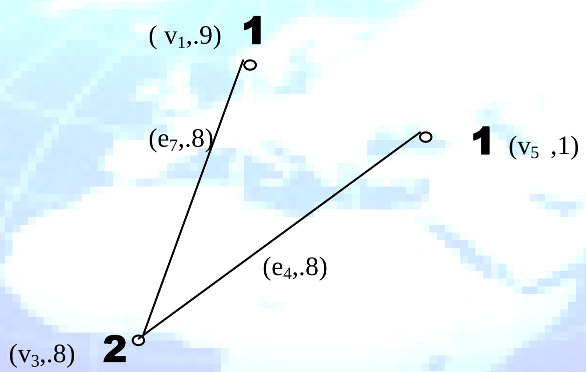


Figure 5

For $\alpha=.9$, Fuzzy graph $G=(V,\sigma,\mu)$ where $\sigma=\{0.9,1\}$ and edge membership value is

$$\mu = \begin{array}{c} \\ v_1 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cc} v_1 & v_5 \end{array} \\ \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \end{array}$$

we find the graph $G_{.9}$ =($V_{.9}$, $E_{.9}$) where $V_{.9}$ ={ v_1 , v_5 } and $E_{.9}$={ }Here we need minimum 1color to proper color all the vertices of the graph $G_{.9}$. so the chromatic number of $G_{.9}$ is 1.

For α= .9 $\Rightarrow$ $\chi_{.9} = \chi(G_{.9})=1$

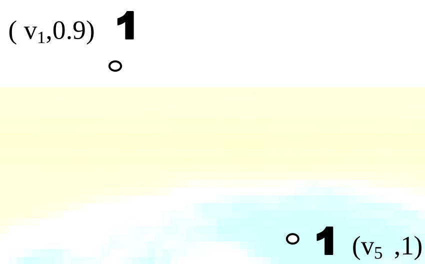


Figure 6

For α= 1, Fuzzy graph G =(V,σ,μ) has σ={1} and we find the graph G_1 =(V_1, E_1) where V_1 ={ v_5 } and E_1={ }Here we need minimum 1color to proper color all the vertices of the graph G_1 . so the chromatic number of G_1 is 1.

For α= 1 $\Rightarrow$ $\chi_1 = \chi(G_1)=1$

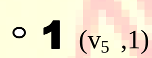


Figure 7

In the above example, six crisp graph G_α =(V_α , E_α) are obtained by considering value the values of α. Now for the chromatic number χ_α for any α,it can be shown that the chromatic number of fuzzy graph G is $\chi(G)=\{(5,0.5),(3,0.6),(3,0.7),(2,0.8),(1,0.9),(1,1)\}$

4. Conclusion and scope of future work

In this paper, we compute fuzzy chromatic number based on α cut of a fuzzy graph whose edge and vertices both are fuzzy set. Here chromatic number of fuzzy graph will be decrease when the value of α cut of the fuzzy graph will increase. In our next paper, we try to apply this approach in total coloring of fuzzy graph.

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